

INTERPOLATION SCHEMES IN WEIGHTED BERGMAN SPACES

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ABSTRACT. We extend our development of interpolation schemes in [3] to more general weighted Bergman spaces.

1. INTRODUCTION

Let A denote area measure and let G be a domain in the complex plane. Let $\mathcal{H}(G)$ denote the space of holomorphic functions on G and $L^p(G) = L^p(G, dA)$ the usual Lebesgue space of measurable functions f with $\|f\|_{p,G}^p = \int_G |f|^p dA < \infty$. The *Bergman space* $A^p(G)$ is $L^p(G) \cap \mathcal{H}(G)$, a closed subspace of $L^p(G)$. If $1 \leq p < \infty$, $A^p(G)$ are Banach spaces and if $0 < p < 1$ they are quasi-Banach spaces. We will allow all $0 < p < \infty$ and abuse the terminology by calling $\|\cdot\|_{p,G}$ a norm even when $p < 1$. In the case where $G = \mathbb{D}$, the open unit disk, we will abbreviate: $L^p = L^p(\mathbb{D})$, $A^p = A^p(\mathbb{D})$ and $\|\cdot\|_p = \|\cdot\|_{p,\mathbb{D}}$.

Let $\psi(z, \zeta)$ denote the *pseudohyperbolic metric*:

$$\psi(z, \zeta) = \left| \frac{z - \zeta}{1 - \bar{\zeta}z} \right|.$$

We will use $D(z, r)$ for the *pseudohyperbolic disk* of radius r centered at z , that is, the ball of radius $r < 1$ in the pseudohyperbolic metric. Let $d\lambda(z) = (1 - |z|^2)^{-2} dA(z)$ denote the *invariant area measure* on \mathbb{D} .

We abbreviate derivatives $\partial/\partial z$ and $\partial/\partial \bar{z}$ by ∂ and $\bar{\partial}$ and the combination $\partial\bar{\partial}u$ will be called the *Laplacian of u* . The *invariant Laplacian* of u , denoted $\tilde{\Delta}u$, is defined by $\tilde{\Delta}u(z) = (1 - |z|^2)^2 \partial\bar{\partial}u(z)$.

Let φ be a C^2 function in \mathbb{D} satisfying $0 < m \leq \tilde{\Delta}\varphi(z) \leq M < \infty$, for positive constants m and M . We define the weighted Bergman space A_φ^p to consist of all functions f that are analytic in \mathbb{D} and satisfy the following

$$(1.1) \quad \|f\|_{\varphi,p} = \left(\int_{\mathbb{D}} \frac{|f(z)e^{-\varphi(z)}|^p}{1 - |z|^2} dA(z) \right)^{1/p} < \infty$$

With $p = 2$ only, these spaces were considered by A. Schuster and T. Wertz in [5] (our formulation differs by a factor of 2 in φ). In that paper, a necessary condition was obtained for a certain weighted interpolation problem they called *O-interpolation* (presumably after its origins in a paper by S. Ostrovsky [4]).

The purpose of this paper is to extend the current author's results in [3] to these more general weighted Bergman spaces, and as a consequence to extend the results of [5] to $p \neq 2$.

Following [3], we define an *interpolation scheme* \mathcal{I} to consist of connected open sets $G_k \subset \mathbb{D}$, $k = 1, 2, 3, \dots$ and corresponding disjoint finite nonempty multisets $Z_k \subset G_k$ (*multisets* are sets with multiplicity) satisfying the following

- (a) there exists $\epsilon > 0$ such that $(Z_k)_\epsilon \subset G_k$ for every k , and

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- (b) there exists $0 < R < 1$ such that for every k the pseudohyperbolic diameter of G_k is no more than R .

The notation $(S)_\epsilon$ for a subset $S \subset \mathbb{D}$ denotes the ϵ -neighborhood of S (in the pseudohyperbolic metric), and the *pseudohyperbolic diameter* of a set $S \subset \mathbb{D}$ is $\sup\{\psi(z, w) : z, w \in S\}$. We remark that G_k are not required to be disjoint. They are also not required to be simply connected, but it is no real loss of generality to assume that they are, or even to assume that G_k are pseudohyperbolic disks of constant radius.

Since finite sets are trivial for our problem, we will always assume the number of clusters is countably infinite.

Given a pair (G_k, Z_k) in an interpolation scheme \mathcal{I} , let \mathcal{N}_k consist of all functions in $\mathcal{H}(G_k)$ (holomorphic on G_k) that vanish on Z_k with the given multiplicities. An interpolation problem can be thought of as specifying values for f and its derivatives at the points of $\mathcal{Z} = \bigcup Z_k$, but it could equally well be thought of as specifying functions $g_k \in \mathcal{H}(G_k)$ and requiring $g_k - f|_{G_k} \in \mathcal{N}_k$. That is, we consider certain sequences (w_k) where each w_k is a coset of \mathcal{N}_k in $\mathcal{H}(G_k)$ and then we say that f *interpolates* (w_k) if, for each k , $f|_{G_k} \in w_k$. Simple interpolation corresponds to the case where each Z_k is a singleton $\{z_k\}$. Then the quotient space $\mathcal{H}(G_k)/\mathcal{N}_k$ is one dimensional and each coset is determined by the common value of its members at z_k .

Given this point of view we need to provide an appropriately normed sequence space and define our interpolation problem. We suppress the dependence on p and φ in the notation and define the sequence space $X_{\mathcal{I}}$ to consist of all sequences $w = (w_k)$ where $w_k \in E_K = \mathcal{H}(G_k)/\mathcal{N}_k$ and $\|w\| = (\sum \|w_k\|^p)^{1/p} < \infty$, where the norm of the coset w_k is the quotient norm:

$$(1.2) \quad \|w_k\|^p = \inf \left\{ \int_{G_k} \frac{|g(z)e^{-\varphi(z)}|^p}{1 - |z|^2} dA(z) : g \in w_k \right\}$$

Since every coset of \mathcal{N}_k contains a polynomial, the norms $\|w_k\|$ are finite. It is not hard, especially in light of later results, to see that in the case of singleton Z_k this is equivalent to a space consisting of sequences of constants (c_k) satisfying

$$\|(c_k)\|^p = \sum |c_k|^p e^{-p\varphi(z_k)} (1 - |z_k|^2) < \infty.$$

Now we can define the interpolation problem and interpolating sequences. The interpolation problem is the following: given a sequence $(w_k) \in X_{\mathcal{I}}$, find a function $f \in A_{\varphi}^p$ such that $f|_{G_k} \in w_k$ for every k . Since a coset w_k can be represented by a function g_k on G_k with norm arbitrarily close to that of w_k , we could equally well describe the problem by: given analytic functions g_k on G_k for each k , satisfying

$$\sum_k \int_{G_k} \frac{|g_k(z)e^{-\varphi(z)}|^p}{1 - |z|^2} dA(z) < \infty$$

find $f \in A_{\varphi}^p$ such that $g_k - f|_{G_k} \in \mathcal{N}_k$.

We say $\mathcal{Z} = \bigcup Z_k$ is an *interpolating sequence relative to the scheme \mathcal{I}* if every such interpolation problem has a solution.

That is, if we define the *interpolation operator* Φ by letting $\Phi(f)$ be the sequence of cosets $(f|_{G_k} + \mathcal{N}_k)$, then an interpolating sequence is one where $\Phi(A_{\varphi}^p)$ contains $X_{\mathcal{I}}$. At the moment, we do not require that Φ take A_{φ}^p into $X_{\mathcal{I}}$, but we will see that it does in fact do so, and is a bounded linear mapping.

One important step will be to show that if \mathcal{Z} is an interpolating sequence relative to a scheme \mathcal{I} then the scheme must satisfy two additional properties: (1) there is a positive lower bound on the distance between different Z_k and (2) there is an upper bound on the

cardinality of the Z_k . Schemes satisfying these two properties will be called *admissible*, and our main theorem will be that a sequence is interpolating relative to an admissible scheme if and only if it satisfies a density inequality we will define later. An important property of this result is that the density inequality depends only on the sequence \mathcal{Z} and not on the scheme itself. That is why we apply the adjective ‘interpolating’ to \mathcal{Z} rather than the scheme. Also, once this has been established, the qualification ‘relative to \mathcal{I} ’ will become redundant.

2. PRELIMINARY RESULTS

It may not be immediately obvious that A_φ^p is nontrivial. This will follow from the following two results.

Lemma 2.1. *Let φ be subharmonic and suppose there exist constants $0 < m \leq M < \infty$ such that $m \leq \tilde{\Delta} \varphi(z) \leq M$ for all $z \in \mathbb{D}$. If $\alpha > 0$ and we set*

$$\tau(z) = \varphi(z) - \alpha \log \left(\frac{1}{1 - |z|^2} \right)$$

then

$$m - \alpha \leq \tilde{\Delta} \tau(z) \leq M - \alpha \quad \text{and} \quad \frac{e^{-p\varphi(z)}}{1 - |z|^2} = e^{-p\tau(z)}(1 - |z|^2)^{\alpha p - 1}$$

The proof is an obvious computation. Since τ satisfies the same condition as φ if α is chosen with $\alpha < m$, the set of spaces A_φ^p (ranging over all such φ) are the same as set the spaces $A_\varphi^{p,\alpha}$ (ranging over all such φ and all $\alpha > 0$), whose norms are defined by

$$\|f\|_{p,\varphi,\alpha} = \left(\int_{\mathbb{D}} |f(z) e^{-\varphi(z)}|^p (1 - |z|^2)^{\alpha p - 1} dA(z) \right)^{1/p}$$

The following was proved in [3] and also in [5] (stated somewhat differently and with a somewhat different proof).

Lemma 2.2. *Let φ be subharmonic and assume $\tilde{\Delta} \varphi$ is bounded. Then there exists a constant C and, for each $a \in \mathbb{D}$, a harmonic function h_a such that the difference $\tau_a = \varphi - h_a$ satisfies*

- (a) $\tau_a(z) \geq 0$ for all $z \in \mathbb{D}$,
- (b) $\tau_a(a) \leq C \|\tilde{\Delta} \varphi\|_\infty$, and
- (c) $\|(1 - |z|^2) \bar{\partial} \tau_a(z)\|_\infty \leq C \|\tilde{\Delta} \varphi\|_\infty$.

The last statement in the lemma was not mentioned in [3], but comes out of the integral formula for $\varphi(z) - h_0(z)$: differentiate under the integral sign and apply standard estimates. It happens that C does not depend on φ , but it is more important that it does not depend on $a \in \mathbb{D}$.

The gradient inequality (c) implies the following.

Lemma 2.3. *With the same hypotheses as Lemma 2.2 and the same h_a , let $0 \leq R < 1$. Then $\varphi(z) - h_a(z)$ is Lipschitz in the hyperbolic metric (with Lipschitz constant a multiple of $\|\tilde{\Delta} \varphi\|_\infty$), and therefore there exists C_R such that $\varphi(z) - h_a(z) \leq C_R \|\tilde{\Delta} \varphi\|_\infty$ for all $z \in D(a, R)$.*

Note that Lemma 2.2 allows us to write the norm of a function in $A_\varphi^{p,\alpha}$ as follows, where we let $H(z)$ be an analytic function in \mathbb{D} with $\operatorname{Re} H(z) = h_0(z)$

$$\int_{\mathbb{D}} |f(z) e^{-H(z)} e^{-\varphi(z) + h_0(z)}|^p (1 - |z|^2)^{\alpha p - 1} dA(z)$$

The exponent $-\varphi(z) + h_0(z)$ is negative, so that exponential is bounded. Moreover, the function $(1 - |z|^2)^{\alpha p - 1}$ is integrable. Thus $A_\varphi^{p,\alpha}$ contains all bounded multiples of $\exp(H(z))$ and so is certainly a nontrivial space.

It is easy to see that these transformations of φ (adding a multiple of $\log(1 - |z|^2)$ and subtracting the harmonic function h_0) convert the original interpolation problem into an equivalent one. Thus, it is without loss of generality that we can assume φ already has the properties of $\varphi - h_0$ in the above lemma. Therefore, the rest of this paper will be concerned with the following reduction of the interpolation problem.

The function φ is *positive* and subharmonic, and there exist constants m, M such that $0 < m \leq \tilde{\Delta}\varphi(z) < M < \infty$ for all $x \in \mathbb{D}$. Moreover $(1 - |z|^2)\bar{\partial}\varphi(z)$ is bounded. Let $\mathcal{I} = \{(G_k, Z_k), k = 1, 2, 3, \dots\}$ be an interpolation scheme and let $p > 0$ and $\alpha > 0$. For a coset $w_k \in \mathcal{H}(G_k)/\mathcal{N}_k$ define its norm $\|w_k\|$ by

$$\|w_k\|^p = \inf \left\{ \int_{G_k} \left| g(z) e^{-\varphi(z)} \right|^p (1 - |z|^2)^{\alpha p - 1} dA(z) : g \in w_k \right\}$$

Given a sequence of cosets (w_k) satisfying $\sum_k \|w_k\|^p < \infty$, the interpolation problem is to find $f \in A_\varphi^{p,\alpha}$ such that $f|_{G_k} \in w_k$, (i.e., f *interpolates* (w_k)). The sequence $\mathcal{Z} = \bigcup Z_k$ is called an *interpolating sequence* for $A_\varphi^{p,\alpha}$ if every such interpolation problem has a solution.

3. PROPERTIES OF INTERPOLATING SEQUENCES

Here we present several properties of interpolating sequences. These are the same as the corresponding results in [3] and the proofs are, for the most part, the same. Therefore I will only indicate how a proof differs in those cases where it does.

The first is that interpolating sequences are zero sequences. We use $\mathcal{Z}(f)$ to denote the multiset (set with multiplicity) of zeros of f .

Proposition 3.1. *Given an interpolation scheme \mathcal{I} with domains G_k and clusters \mathcal{Z}_k , if $\mathcal{Z} = \bigcup_k \mathcal{Z}_k$ is an interpolating sequence for $A_\varphi^{p,\alpha}$, then there is a function $f \in A_\varphi^{p,\alpha}$ such that $\mathcal{Z}(f) = \mathcal{Z}$.*

The only thing we need that is different from the proof in [3] is a different reference for the fact that a subsequence of an $A_\varphi^{p,\alpha}$ -zero sequence is also an $A_\varphi^{p,\alpha}$ -zero sequence. This follows from [1], especially section 5 where weighted spaces of the type considered here are covered.

Theorem 3.2. *Given an interpolation scheme \mathcal{I} with clusters \mathcal{Z}_k , if $\mathcal{Z} = \bigcup_k \mathcal{Z}_k$ is an interpolating sequence for $A_\varphi^{p,\alpha}$ then there is a lower bound $\delta > 0$ on the pseudohyperbolic distance between different clusters of \mathcal{I} .*

The proof in [3] makes use of the following inequality

$$|f'(z)(1 - |z|^2)|^p \leq \frac{C_r}{|D(z, r)|} \int_{D(z, r)} |f(w)|^p dA(w)$$

From this we can deduce that

$$\left| f'(z)(1 - |z|^2) e^{-\varphi(z)} \right|^p (1 - |z|^2)^{\alpha p - 1} \leq \frac{C_r}{|D(z, r)|} \int_{D(z, r)} \left| f(w) e^{-\varphi(w)} \right|^p (1 - |w|^2)^{\alpha p - 1} dA(w)$$

using the inequality of Lemma 2.3. After that, the proof is the same.

I should add that a similar inequality for $f(z)$:

$$\left| f(z) e^{-\varphi(z)} \right|^p (1 - |z|^2)^{\alpha p - 1} \leq \frac{C_r}{|D(z, r)|} \int_{D(z, r)} \left| f(w) e^{-\varphi(w)} \right|^p (1 - |w|^2)^{\alpha p - 1} dA(w)$$

shows that the unit ball of $A_\varphi^{p,\alpha}$ is a normal family and therefore these spaces are complete.

In [3], part of the definition of an interpolating sequence was that the interpolation operator was bounded. We have not made that assumption here. Thus we cannot use the open mapping principle to obtain an *interpolation constant*. We nevertheless obtain one as follows.

Let $I_{\mathcal{Z}}$ consist of all functions in $A_\varphi^{p,\alpha}$ that vanish on \mathcal{Z} with at least the given multiplicities. Then for any interpolation scheme \mathcal{I} for which \mathcal{Z} is an interpolating sequence, there is a map from $X_{\mathcal{I}}$ to the quotient space $A_\varphi^{p,\alpha}/I_{\mathcal{Z}}$ taking a sequence of cosets (w_k) to the coset of functions that interpolate it. It is straightforward to see that this mapping has closed graph and, since both $X_{\mathcal{I}}$ and $A_\varphi^{p,\alpha}/I_{\mathcal{Z}}$ are complete, it is continuous. If K is the norm of this mapping, then every sequence $w \in X_{\mathcal{I}}$ is interpolated by a coset in $A_\varphi^{p,\alpha}/I_{\mathcal{Z}}$ with quotient norm at most $K\|w\|$. By a normal families argument, we can select a representative function (i.e., an element of the same coset) also with norm at most $K\|w\|$. The minimal K for which this is satisfied is called the *interpolation constant* for \mathcal{Z} (relative to the scheme \mathcal{I}).

Theorem 3.2 implies that if \mathcal{Z} is interpolating relative to an interpolation scheme, then the sets G_k have bounded overlap. That is, for some constant M we have $\sum \chi_{G_k}(z) \leq M$ for all $z \in \mathbb{D}$. (See [3] for the details.) Now, every function $f \in A_\varphi^{p,\alpha}$ defines a sequence of cosets (w_k) , where w_k is the coset determined by $f|_{G_k}$. We can therefore estimate the norm of each coset by

$$\|w_k\|^p \leq \int_{G_k} \left| f(z) e^{-\varphi(z)} \right|^p (1 - |z|^2)^{\alpha p - 1} dA(z)$$

Summing these and using the bounded overlap, we get

$$\sum \|w_k\|^p \leq M \int_{\mathbb{D}} \left| f(z) e^{-\varphi(z)} \right|^p (1 - |z|^2)^{\alpha p - 1} dA(z)$$

That is, $(w_k) \in X_{\mathcal{I}}$. Thus, for the mapping Φ taking each f to its sequence of cosets we have not only $X_{\mathcal{I}} \subset \Phi(A_\varphi^{p,\alpha})$, but also $\Phi(A_\varphi^{p,\alpha}) \subset X_{\mathcal{I}}$ and Φ is bounded.

If \mathcal{I} and \mathcal{I}' are interpolation schemes, we will say that \mathcal{I}' is a *subscheme* of \mathcal{I} if for each pair (G'_k, Z'_k) of \mathcal{I}' there exists a pair (G_k, Z_k) of \mathcal{I} such that $G'_k = G_k$ and Z'_k is a subset (with multiplicity) of Z_k .

Proposition 3.3. *If $\mathcal{Z} = \bigcup Z_k$ is an interpolating sequence for $A_\varphi^{p,\alpha}$ relative to the interpolation scheme $\mathcal{I} = \{(G_k, Z_k), k = 1, 2, 3, \dots\}$ and if $\mathcal{I}' = \{(G'_k, Z'_k), k = 1, 2, 3, \dots\}$ is a subscheme, then $\mathcal{Z}' = \bigcup Z'_k$ is an interpolating sequence for $A_\varphi^{p,\alpha}$ relative to \mathcal{I}' . The interpolation constant for \mathcal{I}' is less than or equal to the constant for \mathcal{I} .*

The proof is the same as in [3].

Invariance under Möbius transformations is just slightly more involved, because composition will also change the function φ . However, the new function will satisfy the same conditions as φ . We will normally want, after composition, the new weight to remain bounded above and also satisfy a uniform lower bound at 0. Thus, given a point $a \in \mathbb{D}$, let $M_a(z) = (a - z)/(1 - \bar{a}z)$, a Möbius transformation that maps a to 0 and is its own inverse. Given a space $A_\varphi^{p,\alpha}$, let $\varphi_a(z) = \varphi(M_a(z)) - h_a(M_a(z))$, where h_a is the harmonic function of Lemma 2.2.

Proposition 3.4. *Let \mathcal{I} be an interpolation scheme with clusters Z_k and domains G_k . If $\mathcal{Z} = \bigcup Z_k$ is interpolating for $A_\varphi^{p,\alpha}$ with respect to \mathcal{I} and $a \in \mathbb{D}$, then $M_a(\mathcal{Z})$ is interpolating for $A_{\varphi_a}^{p,\alpha}$ relative to the scheme $M_a(\mathcal{I})$ which has clusters $M_a(Z_k)$ and domains $M_a(G_k)$. Moreover, the interpolation constants are the same.*

Proof. The map $\Phi_a f = (f e^{-H_a}) \circ M_a (M'_a)^{\alpha+1/p}$ (where H_a is chosen with $\operatorname{Re} H_a = h_a$ and say $\operatorname{Im} H_a(a) = 0$) is an isometry from $A_{\varphi}^{p,\alpha}$ to $A_{\varphi_a}^{p,\alpha}$. It maps the \mathcal{N}_k associated with Z_k to the \mathcal{N}'_k associated with Z'_k and therefore maps a coset w_k of \mathcal{N}_k to a coset w'_k of \mathcal{N}'_k . Moreover, the mapping of cosets is isometric. Thus, Φ_a converts any interpolation problem for $A_{\varphi_a}^{p,\alpha}$ to an isometric problem for $A_{\varphi}^{p,\alpha}$ and the inverse converts its solution to an isometric solution. \square

One key requirement of an interpolating sequence is that adding a single point to it produces an interpolating sequence (for an appropriately augmented scheme), with a suitable estimate on the new interpolation constant.

Proposition 3.5. *Let \mathcal{I} be an interpolation scheme with clusters Z_k and domains G_k . Suppose $\mathcal{Z} = \bigcup Z_k$ is an interpolating sequence for $A_{\varphi}^{p,\alpha}$ relative to \mathcal{I} and let $z_0 \in \mathbb{D}$. Suppose there is an $\epsilon > 0$ such that $\psi(z_0, Z_k) > \epsilon$ for every k . Define a new scheme \mathcal{J} whose domains are all the domains of \mathcal{I} plus the domain $G_0 = D(z_0, 1/2)$ and whose clusters W_k are all the Z_k plus $W_0 = \{z_0\}$. Then $\mathcal{W} = \{z_0\} \cup \mathcal{Z}$ is an interpolating sequence relative to the scheme \mathcal{J} .*

If K is the interpolation constant for \mathcal{Z} then the constant for \mathcal{W} is at most CK/ϵ , where C is a positive constant that depends only on the space $A_{\varphi}^{p,\alpha}$.

Proof. Some of the proof in [3] is simplified by the symmetry of the weights, so we will have to add a little detail. Without loss of generality we may assume $z_0 = 0$. Suppose we wish to interpolate a sequence of cosets $w = (w_k, k = 0, 1, 2, \dots)$ in $X_{\mathcal{J}}$, with $\|w\| = 1$. Choose representative functions g_k of minimal norm for all w_k with $k \neq 0$. For $k = 0$, w_0 contains a constant function g_0 . This may not be the minimizing representative, but from the inequality

$$\left| f(0) e^{-\varphi(0)} \right|^p \leq C \int_{G_0} \left| f(z) e^{-\varphi(z)} \right|^p (1 - |z|^2)^{\alpha p - 1} dA(z)$$

we can estimate $\|w_0\|$ within a constant factor by using g_0 .

Now consider the functions $f_k = (g_k - g_0)/z$ for $k \neq 0$. One easily estimates

$$\int_{G_k} \left| f_k(z) e^{-\varphi(z)} \right|^p (1 - |z|^2)^{\alpha p - 1} dA(z) \leq \frac{C_p}{\epsilon^p} (\|w_k\|^p + |g_0|^p \mu(G_k))$$

Where μ is the measure $e^{-p\varphi(z)}(1 - |z|^2)^{\alpha p - 1} dA(z)$. Therefore the sequence of cosets (u_k) represented by (f_k) belongs to $X_{\mathcal{I}}$, having norm at most $C_p^{1/p}(1 + C\mu(\mathbb{D}))/\epsilon$ for some C .

Since \mathcal{Z} is interpolating, there exist $f \in A_{\varphi}^{p,\alpha}$ that interpolates (u_k) with norm at most $K\|(u_k)\| = CK/\epsilon$ for some constant. Then $zf(z) + g_0$ interpolates (w_k) with norm at most CK/ϵ for some other constant C . \square

We say that \mathcal{Z} has bounded density if for $0 < R < 1$ there is a finite constant $N = N_R$ such that every disk $D(a, R)$, $a \in \mathbb{D}$, contains no more than N points (counting multiplicity). If there is a finite upper bound for some $R \in (0, 1)$ then there is a finite upper bound for any $R \in (0, 1)$, although the bounds will be different. We will show that an interpolating sequence relative to a scheme \mathcal{I} must have bounded density. Given the bounded overlap of the domains and the uniform separation between clusters, it is enough to show that there is an upper bound on the number of points in each cluster (counting multiplicity).

Theorem 3.6. *If \mathcal{Z} is an interpolating sequence for $A_{\varphi}^{p,\alpha}$ relative to an interpolation scheme \mathcal{I} then there is a finite upper bound B on the number of points, counting multiplicity, in each cluster Z_k of \mathcal{I} .*

The proof is the same as in [3] except we use Möbius transformations M_a to map $A_\varphi^{p,\alpha}$ to $A_{\varphi_a}^{p,\alpha}$ as before. It is important that there is a lower bound on $\varphi_a(0)$ independent of a . This means there is also a lower bound on φ_a on compact sets, allowing the normal families argument to proceed.

As in [3], we now have two additional conditions that the scheme \mathcal{I} must satisfy in order for the sequence $\mathcal{Z} = \bigcup Z_k$ to be interpolating, and we call such schemes admissible.

Summarizing, we have defined $\mathcal{I} = \{(G_k, Z_k), k = 1, 2, 3, \dots\}$ to be an interpolation scheme if it satisfies properties P1 and P2 below. We will say \mathcal{I} is an *admissible interpolation scheme* if it also satisfies P3 and P4:

- (P1) There is an $R < 1$ such that the pseudohyperbolic diameter of each G_k is at most R .
- (P2) There is an $\epsilon > 0$ such that $(Z_k)_\epsilon \subset G_k$ for every k .
- (P3) There is a $\delta > 0$ such that for all $j \neq k$ the pseudohyperbolic distance from Z_j to Z_k is at least δ .
- (P4) There is an upper bound B on the number of points (counting multiplicity) in each cluster Z_k .

As in [3], any sequence \mathcal{Z} with bounded density can be subdivided into clusters Z_k , with associated open sets G_k , so that the result is an admissible interpolation scheme. It will not be needed, but it may be interesting that the scheme produced satisfies $G_k = (Z_k)_\epsilon$ for some $\epsilon > 0$, and moreover the G_k are disjoint. One could therefore ‘fill in the holes’ and have a scheme with simply connected domains.

4. ZERO SETS, DENSITY, AND THE $\bar{\partial}$ -PROBLEM

The following perturbation result differs little in proof from the version in [3]. The phrase *interpolation invariants* means quantities, such as the interpolation constant, that are unchanged under a Möbius transformation of the disk. This includes the numbers p and α and in this paper also the estimates on $\tilde{\Delta}\varphi$.

Proposition 4.1. *Let \mathcal{I} be an admissible interpolation scheme with domains G_k and clusters Z_k . Assume $\mathcal{Z} = \bigcup_k Z_k$ is an interpolating sequence for $A_\varphi^{p,\alpha}$ with interpolation constant K . For each k let β_k be defined by $\beta_k(z) = r_k z$ and let \mathcal{J} be the interpolation scheme with domains $D_k = \beta_k(G_k)$ and clusters $W_k = \beta_k(Z_k)$. Let $\mathcal{W} = \bigcup_k W_k$.*

There exists an $\eta > 0$ depending only on interpolation invariants such that if $\psi(\beta_k(z), z) < \eta$ for all $z \in G_k$ and for all k , then \mathcal{W} is an interpolating sequence for $A_\varphi^{p,\alpha}$ relative to \mathcal{J} . Its interpolation constant can be estimated in terms of η and interpolation invariants of \mathcal{I} .

One stage in the proof in [3] is an estimate of $|f(z/r_k) - f(z)|^p$ by a small multiple of the average of $|f|^p$ on $(D_k)_{1/2}$ (the new domains expanded by pseudohyperbolic distance $1/2$). This particular step can be done similarly when weighting with $e^{-p\varphi}(1 - |z|^2)^{\alpha p - 1}$. This relies mostly on the fact that φ is Lipschitz. The rest of the proof is essentially the same.

In [1] it was shown that the following function could be used to determine whether a sequence \mathcal{Z} in \mathbb{D} is a zero sequence for a variety of analytic function spaces:

$$k_{\mathcal{Z}}(\zeta) = \sum_{a \in \mathcal{Z}} k_a(z) = \sum_{a \in \mathcal{Z}} \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^2} \frac{|z|^2}{2}$$

where a point with multiplicity m occurs m times in the sum. In particular, \mathcal{Z} is a zero set if and only if a certain weighted function space is nontrivial. In our current context (covered in the last section of [1]), we have the following theorem.

Theorem 4.2. *Let \mathcal{Z} be a sequence in \mathbb{D} . Define the function $k_{\mathcal{Z}}$ as above. The following are equivalent.*

- (a) \mathcal{Z} is a zero set for some function in $A_{\varphi}^{p,\alpha}$.
- (b) There exists a **nowhere zero** analytic function F such that

$$(4.1) \quad \int_{\mathbb{D}} \left| F(\zeta) e^{-\varphi(\zeta)} \right|^p e^{pk_{\mathcal{Z}}(\zeta)} (1 - |\zeta|^2)^{\alpha p - 1} dA(\zeta) < \infty$$

- (c) There exists a **nonzero** analytic function F satisfying (4.1).

The integral in (4.1) defines a norm that determines a space we will call $A_{\varphi, \mathcal{Z}}^{p,\alpha}$. Then \mathcal{Z} is a zero set for $A_{\varphi}^{p,\alpha}$ if and only if $A_{\varphi, \mathcal{Z}}^{p,\alpha}$ is non trivial.

Moreover, if we define

$$\Psi_{\mathcal{Z}}(\zeta) = z^m \prod_{\substack{a \in \mathcal{Z} \\ a \neq 0}} \bar{a} \frac{a - z}{1 - \bar{a}z} \exp \left(1 - \bar{a} \frac{a - z}{1 - \bar{a}z} \right)$$

(where m is the multiplicity of the origin, if it belongs to \mathcal{Z} , and zero otherwise) then $f \mapsto f/\Psi_{\mathcal{Z}}$ is a one-to-one correspondence between functions in $A_{\varphi}^{p,\alpha}$ that vanish on \mathcal{Z} to at least the given multiplicities and $A_{\varphi, \mathcal{Z}}^{p,\alpha}$. We will be applying this only when \mathcal{Z} has no points in $D(0, \delta)$ for some fixed $\delta > 0$, in which case the value of $|\Psi_{\mathcal{Z}}(0)| = \prod |a|^2 e^{1-|a|^2}$ can be estimated from below in terms of δ and the density of \mathcal{Z} .

Note that the convergence of the product defining $\Psi_{\mathcal{Z}}$ requires the sequence $(1 - |a|^2)^2$, $a \in \mathcal{Z}$, to be summable. This follows from the formula (1) in [1] in light of the discussion in section 5 of that paper. For interpolating sequences, which have bounded density, this is automatically true without any need for the results in [1].

In the case where $\varphi \equiv 0$, the paper [3] showed that \mathcal{Z} is an interpolating sequence if and only a certain density condition is satisfied. In the general case, that density condition will involve integrals of φ . It was also shown that this is equivalent to bounds on the solutions u of the $\bar{\partial}$ -equation

$$(1 - |z|^2) \bar{\partial} u = f$$

in a certain weighted function space. In the general case let $L_{\varphi, \mathcal{Z}}^{p,\alpha}$ be the measurable function version of $A_{\varphi, \mathcal{Z}}^{p,\alpha}$. We need a bounded operator on this space that maps f to a solution u .

Theorem 4.3. *Let \mathcal{Z} be a set with multiplicity in \mathbb{D} , $p \geq 1$, $\alpha > 0$, and φ a positive subharmonic function satisfying $0 < m \leq \tilde{\Delta} \varphi < M < \infty$ in \mathbb{D} . The following are equivalent:*

- (a) \mathcal{Z} is an interpolating sequence for $A_{\varphi}^{p,\alpha}$ relative to any admissible interpolation scheme.
- (b) \mathcal{Z} is an interpolating sequence for $A_{\varphi}^{p,\alpha}$ relative to some interpolation scheme.
- (c) The upper uniform density $S_{\varphi}^{+}(\mathcal{Z})$ (defined below) is less than α .
- (d) \mathcal{Z} has bounded density and the $\bar{\partial}$ -problem has a bounded solution operator on $L_{\varphi, \mathcal{Z}}^{p,\alpha}$.

We postpone the proof to discuss the density condition. We prefer to use the following summation to define density. It was shown in [2] to be equivalent to the usual one for the standard weights.

For $r \in (0, 1)$ let

$$\hat{k}_{\mathcal{Z}}(r) = \frac{1}{2\pi} \int_0^{2\pi} k_{\mathcal{Z}}(re^{it}) dt = \frac{r^2}{2} \sum_{a \in \mathcal{Z}} \frac{(1 - |a|^2)^2}{1 - |a|^2 r^2}$$

then let

$$S(\mathcal{Z}, r) = \frac{\hat{k}_{\mathcal{Z}}(r)}{\log\left(\frac{1}{1-r^2}\right)}$$

For each $a \in \mathbb{D}$, let $\mathcal{Z}_a = M_a(\mathcal{Z})$, where as before M_a is the Möbius transformation exchanging a and 0. In case $\varphi \equiv 0$, the density we used in [3] was $S^+(\mathcal{Z})$, defined by

$$S^+(\mathcal{Z}) = \limsup_{r \rightarrow 1-} \sup_{a \in \mathbb{D}} S(\mathcal{Z}_a, r)$$

It was shown in [2] that this is equivalent to the usual *upper uniform density* D^+ for sets \mathcal{Z} (as defined in [6] for example). The density inequality equivalent to interpolation in $A^{p,\alpha}$ (where $\varphi \equiv 0$) is that $S^+(\mathcal{Z}) < \alpha$. (In [2] and [3], the condition was written as $S^+(\mathcal{Z}) < (\alpha + 1)/p$, but the number α there was the exponent of $(1 - |z|^2)$ that we are writing here as $\alpha p - 1$.)

For the more general φ , our density condition has to incorporate φ . Let

$$\hat{\varphi}(r) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(re^{it}) dt - \varphi(0)$$

and define

$$S_{\varphi}(\mathcal{Z}, r) = \frac{\hat{k}_{\mathcal{Z}}(r) - \hat{\varphi}(r)}{\log\left(\frac{1}{1-r^2}\right)}$$

and finally

$$S_{\varphi}^+(\mathcal{Z}) = \limsup_{r \rightarrow 1-} \sup_{a \in \mathbb{D}} S_{\varphi_a}(\mathcal{Z}_a, r)$$

As part of the proof, we need to be able to express this density in terms of the invariant Laplacian of the functions involved. This follows easily from the following, obtained from Green's formula. Recall that $d\lambda(z)$ is the invariant measure $dA(z)/(1 - |z|^2)^2$:

$$\hat{\varphi}(r) = \frac{1}{\pi} \int_{r\mathbb{D}} \tilde{\Delta} \varphi(z) \log\left(\frac{r^2}{|z|^2}\right) d\lambda(z)$$

A similar formula holds for $\hat{k}_{\mathcal{Z}}$. If we combine these two formulas, plus one for $\log\left(\frac{1}{1-|z|^2}\right)$ we get the following formula

$$(4.2) \quad S_{\varphi}(\mathcal{Z}, r) - \alpha = \frac{1}{\pi \log\left(\frac{1}{1-r^2}\right)} \int_{r\mathbb{D}} \tilde{\Delta} \left(k_{\mathcal{Z}}(z) - \varphi(z) - \alpha \log\left(\frac{1}{1-|z|^2}\right) \right) \log \frac{r^2}{|z|^2} d\lambda(z)$$

This relies on the calculation

$$\frac{1}{\pi} \int_{r\mathbb{D}} \log \frac{r^2}{|z|^2} d\lambda(z) = \log\left(\frac{1}{1-r^2}\right).$$

If we temporarily let

$$\begin{aligned} \tau(\zeta) &= k_{\mathcal{Z}}(\zeta) - \varphi(\zeta) - \alpha \log\left(\frac{1}{1-|\zeta|^2}\right) \\ \sigma_r(\zeta) &= \frac{\log \frac{r^2}{|\zeta|^2} \chi_{r\mathbb{D}}(\zeta)}{\pi \log\left(\frac{1}{1-r^2}\right)} \end{aligned}$$

then invariant nature of the formula in (4.2) allows us to write

$$(4.3) \quad S_{\varphi_a}(\mathcal{Z}_a, r) - \alpha = \frac{1}{\pi \log\left(\frac{1}{1-r^2}\right)} \int_{D(a,r)} \tilde{\Delta} \tau(z) \log \frac{r^2}{|M_a(z)|^2} d\lambda(z)$$

and then the right side of equation (4.3) is the *invariant convolution* of $\tilde{\Delta} \tau$ and σ_r . That is

$$S_{\varphi_a}(\mathcal{Z}_a, r) - \alpha = (\tilde{\Delta} \tau) * \sigma_r(a) \equiv \int_{\mathbb{D}} (\tilde{\Delta} \tau(z)) \sigma_r(M_a(z)) d\lambda(z).$$

We know that the invariant convolution has the following properties if one of the functions is radially symmetric (as is σ_r):

$$\begin{aligned} \tau * \sigma_r &= \sigma_r * \tau \\ (\tilde{\Delta} \tau) * \sigma_r &= \tilde{\Delta}(\tau * \sigma_r) \end{aligned}$$

Therefore, the density condition (c) of Theorem 4.3 is equivalent to the requirement that there exists an $r_0 \in (0, 1)$ and an $\epsilon > 0$ such that the invariant Laplacian $\tilde{\Delta}(\tau * \sigma_r)$ is bounded above by $-\epsilon$ for all $r > r_0$. We note that this means we can (and will) invoke Lemma 2.2 on $-(\tau * \sigma_r)$. Note also that the fact that τ is Lipschitz in the hyperbolic metric shows that $\tau - \tau * \sigma_r$ is a bounded function with a bound that depends on r .

Recall that originally the space A_{φ}^p had $\alpha = 0$ and no requirement that φ be positive. We modified it by subtracting $\alpha \log \left(\frac{1}{1-|z|^2} \right)$ and a harmonic function. Consequently, the combination $\varphi(z) + \alpha \log \left(\frac{1}{1-|z|^2} \right)$ that appears in equation (4.3) is in fact the original exponent defining A_{φ}^p , up to an added harmonic function. Therefore the means and invariant Laplacian of $\varphi(z) + \alpha \log \left(\frac{1}{1-|z|^2} \right)$ are the same as those of the original φ .

5. PROOFS

The proof of Theorem 4.3 proceeds just as in [3], so we will only describe the highlights.

Given an interpolating sequence \mathcal{Z} for an admissible scheme \mathcal{I} , we can delete the pairs (G_k, Z_k) where Z_k meets $D(0, 1/2)$ and add the domain $G_0 = D(0, 1/2)$ with cluster $Z_0 = \{0\}$ to obtain a new scheme \mathcal{J} . Then a function f exists with $f(0) = 1$ that vanishes on the union \mathcal{Z}' of the remaining clusters. We get an estimate on the $A_{\varphi}^{p,\alpha}$ -norm of f that depends only on the data about \mathcal{I} that are invariant under Möbius transformations of \mathcal{I} . We can normalize f and then we get a lower bound on the value of $f(0)$. We can modify f so that it vanishes only on \mathcal{Z}' , still having norm 1 and retaining a lower bound on $f(0)$.

We then divide f by $\Psi_{\mathcal{Z}'}$ to get a nonvanishing function in $A_{\varphi, \mathcal{Z}'}^{p,\alpha}$. Since \mathcal{Z} and \mathcal{Z}' differ only in a finite number of points (the number of which can be estimated in terms of interpolation invariants), this space is equivalent to $A_{\varphi, \mathcal{Z}}^{p,\alpha}$. We can do all of this after first perturbing \mathcal{I} inward an amount small enough that the perturbed sequence \mathcal{W} remains an interpolating sequence and so we obtain $f \in A_{\varphi, \mathcal{W}}^{p,\alpha}$ which we normalize to have norm 1 and we still obtain a lower bound on $f(0)$.

Following [3], we can perturb \mathcal{W} back outward to \mathcal{Z} and obtain a constant $\beta < 1$ and a new function g that satisfies

$$\int_{\mathbb{D}} \left| g(z) e^{k_{\mathcal{Z}}(z)} \right|^{p/\beta} e^{-p\varphi(z)} (1 - |z|^2)^{\alpha p - 1} dA = 1$$

while retaining a lower bound on $g(0)$. Solve an extremal problem: maximize $|g(0)|$ subject to the above equality to obtain a new function g such that the above integrand defines a Carleson measure, from which we obtain a constant C such that

$$\left| g(z) e^{k_{\mathcal{Z}}(z)} \right|^{p/\beta} e^{-p\varphi(z)} (1 - |z|^2)^{\alpha p} \leq C \quad \text{for all } z \in \mathbb{D}.$$

Now consider

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} \frac{p}{\beta} \log |g(re^{it})| + \frac{p}{\beta} k_{\mathcal{Z}}(re^{it}) - p(\varphi(re^{it}) - \varphi(0)) + \alpha p \log(1 - |r|^2) dt \\ & \leq \log \left(\frac{1}{2\pi} \int_0^{2\pi} \left| g(re^{it}) e^{k_{\mathcal{Z}}(re^{it})} \right|^{p/\beta} e^{-(\varphi(re^{it}) - \varphi(0))} (1 - |r|^2)^{\alpha p} dt \right) \\ & \leq \log C, \end{aligned}$$

The extra factor $e^{\varphi(0)}$ can be included because we have an estimate on $\varphi(0)$ in terms of $\|\tilde{\Delta} \varphi\|_{\infty}$. We multiply this by β/p and use the fact that the mean of $\log |g|$ exceeds its value at 0 to get

$$\hat{k}_{\mathcal{Z}}(r) - \beta \hat{\varphi} - \beta \alpha \log \frac{1}{1 - r^2} \leq C - \log |g(0)|$$

We can rewrite this in terms of the invariant Laplacian as discussed previously (and incorporate $\log |g(0)|$ into the constant):

$$(5.1) \quad \int_{r\mathbb{D}} \tilde{\Delta} \left(k_{\mathcal{Z}}(r) - \beta \varphi - \beta \alpha \log \frac{1}{1 - |z|^2} \right) \log \left(\frac{r^2}{|z|^2} \right) d\lambda \leq C$$

We can estimate as follows: for some $\epsilon > 0$

$$\beta \tilde{\Delta} \left(\varphi + \alpha \log \left(\frac{1}{1 - |z|^2} \right) \right) \leq \tilde{\Delta} \left(\varphi + (\alpha - 2\epsilon) \log \frac{1}{1 - |z|^2} \right)$$

because the invariant Laplacian on the left side is bounded away from 0 and the invariant Laplacian of the log expression is constant. Inserting this into (5.1) and then rewriting the result in terms of means, we obtain

$$\hat{k}_{\mathcal{Z}}(r) - \hat{\varphi} \leq (\alpha - 2\epsilon) \log \frac{1}{1 - r^2} + C$$

Divide this by $\log \frac{1}{1 - r^2}$ and then, for r sufficiently near 1 we have

$$(5.2) \quad \frac{\hat{k}_{\mathcal{Z}}(r) - \hat{\varphi}}{\log \frac{1}{1 - r^2}} \leq \alpha - \epsilon$$

Since the constants have estimates that are uniform over all Möbius transforms, we can replace \mathcal{Z} by its Möbius transforms \mathcal{Z}_a and take the supremum of the above inequality over all a to obtain the required density condition (c):

$$\sup_{a \in \mathbb{D}} \frac{\hat{k}_{\mathcal{Z}_a}(r) - \hat{\varphi}_a}{\log \frac{1}{1 - r^2}} \leq \alpha - \epsilon$$

for all r sufficiently close to 1.

As we saw at the end of section 4, the condition (5.2) is equivalent to the existence of an negative upper bound on the invariant Laplacian of the convolution $\tau * \sigma_r$ where

$$\begin{aligned} \tau(\zeta) &= k_{\mathcal{Z}}(\zeta) - \varphi(\zeta) - \alpha \log \left(\frac{1}{1 - |\zeta|^2} \right) \\ \sigma_r(\zeta) &= \frac{\log \frac{r^2}{|\zeta|^2} \chi_{r\mathbb{D}}(\zeta)}{\pi \log \left(\frac{1}{1 - r^2} \right)} \end{aligned}$$

Then Lemma 2.2 (applied to $-\tau * \sigma_r(\zeta)$) provides us with a harmonic function h such that $\tau * \sigma_r(\zeta) + h(\zeta)$ is everywhere negative and there is a lower bound on its value at 0 in terms of the sup norm of the invariant Laplacian. Since $\tau - \tau * \sigma_r$ is bounded, we get a similar

result for τ itself. That is, there exists constants C and ϵ (depending only on φ , p , r and the scheme \mathcal{I}) and a harmonic function h such that

$$k_{\mathcal{Z}}(\zeta) - \varphi(\zeta) + h(\zeta) \leq (\alpha - \epsilon) \log \left(\frac{1}{1 - |\zeta|^2} \right)$$

and

$$k_{\mathcal{Z}}(0) - \varphi(0) + h(0) \geq -C$$

Using the uniformity of our estimates over Möbius transformations, we obtain for each $a \in \mathbb{D}$ a harmonic function h_a such that

$$k_{\mathcal{Z}}(\zeta) - \varphi(\zeta) + h_a(\zeta) \leq (\alpha - \epsilon) \log \left(\frac{1}{1 - |M_a(\zeta)|^2} \right)$$

and

$$k_{\mathcal{Z}}(a) - \varphi(a) + h_a(a) \geq -C$$

Exponentiating, we get holomorphic functions $g_a(z)$ and constants $\delta > 0$ and C such that

$$\left| g_a(\zeta) e^{k_{\mathcal{Z}}(\zeta) - \varphi(\zeta)} \right| \leq \frac{1}{(1 - |M_a(\zeta)|^2)^{\alpha - \epsilon}}$$

and

$$\left| g_a(a) e^{k_{\mathcal{Z}}(a) - \varphi(a)} \right| \geq \delta$$

These functions allow us to construct a solution of the $\bar{\partial}$ -equation exactly as in [2]. That is the solution of $(1 - |z|^2) \bar{\partial} u(z) = f(z)$ is given by

$$u(z) = \frac{1}{\pi} \sum_{j=1}^{\infty} g_{a_j}(z) \int_{\mathbb{D}} \frac{\gamma_j(w) f(w)}{g_{a_j}(w)} \frac{(1 - |w|^2)^{m-1}}{(z - w)(1 - \bar{w}z)^m} dA(w)$$

where γ_j is a suitable partition of unity and m is a sufficiently large integer. The lower estimate on $g_{a_j} e^{k_{\mathcal{Z}} - \varphi}$ at a_j allows us to divide by it on the support of γ_j , provided that support is sufficiently small. The upper estimates allow us to show that the operator is bounded on $L_{\varphi, \mathcal{Z}}^{p, \alpha}$. This shows that condition (c) of theorem 4.3 implies condition (d).

Finally, given solutions with bounds for the $\bar{\partial}$ -equation, we can solve any interpolation problem just as in [3]. This ends the (sketch of the) proof.

If one returns to the original space A_{φ}^p , the theorem can be restated as follows:

Theorem 5.1. *Let \mathcal{Z} be a set with multiplicity in \mathbb{D} , $p \geq 1$, and φ a subharmonic function satisfying $0 < m \leq \tilde{\Delta} \varphi < M < \infty$ in \mathbb{D} . The following are equivalent:*

- (a) \mathcal{Z} is an interpolating sequence for A_{φ}^p relative to any admissible interpolation scheme.
- (b) \mathcal{Z} is an interpolating sequence for A_{φ}^p relative to some interpolation scheme.
- (c) $S_{\varphi}^+(\mathcal{Z}) < 0$.
- (d) \mathcal{Z} has bounded density and the $\bar{\partial}$ -problem has a bounded solution operator on $L_{\varphi, \mathcal{Z}}^p$.

6. p LESS THAN 1

Most of the considerations that went into the proof of theorem 4.3 apply equally well to all $p \in (0, \infty)$. However the last step, constructing a solution of the $\bar{\partial}$ -equation, fails when $p < 1$: the integrals in question may not exist when f is not locally integrable. The way around this deficiency is to replace the domain of the $\bar{\partial}$ -equation (normally $L_{\varphi, \mathcal{Z}}^{p, \alpha}$) with

a smaller one. One example: all measurable functions f that are locally in L^q for some $q \in [1, \infty]$ and such that $m_q(f) \in L_{\varphi, \mathcal{Z}}^{p, \alpha}$ where

$$m_q(f)(\zeta) = \begin{cases} \frac{1}{|D(\zeta, 1/2)|} \int_{D(\zeta, 1/2)} |f|^q dA & q < \infty \\ \sup_{z \in D(\zeta, 1/2)} |f(w)| & q = \infty \end{cases}$$

All holomorphic functions wind up in this space, even with $q = \infty$. Moreover, when proving (d) \Rightarrow (a) of theorem 4.3, the function to which one applies the solution operator belongs to this space (even with $q = \infty$). The proof in [2] of the boundedness of this solution works here for $p < 1$ just as well as for $p \geq 1$.

Therefore, Theorem 4.3 is valid for $p < 1$ provided only that in part (d) we replace the space $L_{\varphi, \mathcal{Z}}^{p, \alpha}$ with this modified version.

7. APPLICATION TO O-INTERPOLATION

Let \mathcal{Z} be a sequence of distinct points in \mathbb{D} having bounded density, and let $c_a, a \in \mathcal{Z}$, be sequence of values satisfying

$$(7.1) \quad \sum_{a \in \mathcal{Z}} |c_a|^p \frac{e^{-p\varphi(a)}}{\delta_a^{pn_a}} (1 - |a|^2) < \infty$$

where δ_a is the pseudohyperbolic distance from a to the nearest point in $\mathcal{Z} \setminus \{a\}$ and n_a is the number of points of \mathcal{Z} in $D(a, 1/2)$. Then O-interpolation consists of finding a function $f \in A_{\varphi}^p$ satisfying $f(a) = c_a$ for all $a \in \mathcal{Z}$.

Just as in the addendum to [3] (the last section), we can provide an admissible scheme $\mathcal{I} = \{(G_k, Z_k), k = 1, 2, 3, \dots\}$ for \mathcal{Z} and define functions f_k on G_k that have the values c_a at the points a of \mathcal{Z} that lie in G_k . Moreover, the L^p -norms of these functions provides an upper bound for the norm $\|w_k\|$ of the cosets determined by f_k and these are shown to be less than

$$C \sum_{a \in Z_k} |c_a|^p \frac{e^{-p\varphi(a)}}{\delta_a^{pn_a}} (1 - |a|^2)$$

with C independent of k . Thus the finiteness condition (7.1) dominates $\sum \|w_k\|^p$.

Thus we have created an interpolation problem relative to the scheme \mathcal{I} whose solution would be a function f satisfying $f(a) = c_a$. The density condition now implies that a solution exists in A_{φ}^p . That is, the density condition implies O-interpolation.

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